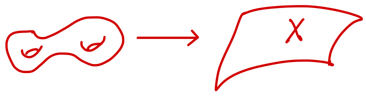
Si Li B-model

2025/3 MIST

- Classical aspects of B-model
- Quantum aspects of B-model
- Open-closed and large N duality
- Worldsheet B-model

§ 1. Introduction

Mirror symmetry : duality of SCFT



Symplectic Geometry $\overset{\text{Fourier}}{\Longleftrightarrow}$ Complex Geometry
(A-model) (B-model)

$$\begin{array}{ccc} \int_{Map(\Sigma,X)} (\text{A-model}) & & \int_{Map(\Sigma,\tilde{X})} (\text{B-model}) \\ \Big\downarrow \text{localize} & & \Big\downarrow \text{localize} \\ \int_{Holom(\Sigma,X)} \sim & & \int_{Const.(\Sigma,\tilde{X})} \sim \end{array}$$

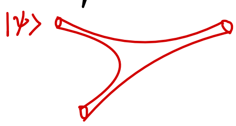
Gromov-Witten type Hodge type theory

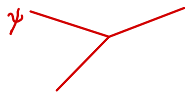
<u>String description</u> (world-sheet theory)

<u>String field theory</u>: describe string in the lagrangian of QFT.

String Fock space $\longrightarrow$ String field
$|\psi\rangle$ ψ

dynamics $\rightsquigarrow$ $S[\psi]$ string field action

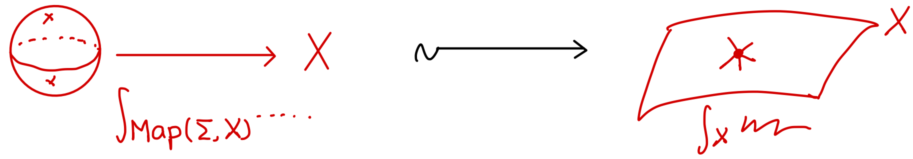




<u>Zwiebach 1992</u> <u>Closed</u> string field theory is described by a string action which contains ∞ number of vertices, s.t. BV master eqt.
In particular, $g=0$ string field $\Longrightarrow$ L_∞-alg.

<u>Zwiebach 1997</u> <u>Open</u> string field via A_∞-alg. & open-closed quantum master eqt. via moduli of bordered Riemann surfaces.

B-model : B-twisted TFT / topo. string
"localized around const. maps" $\rightsquigarrow$ "local" string field action.



<u>Witten 1992</u>: <u>A-model</u> open string field $=$ CS + instantons
<u>B-model</u> open string field $=$ Hol CS on CY3
(and speculate that theory is finite).

<u>BCOV 1994</u> B-model on CY3 as gauge theory. "Kodaira-Spencer gravity"

Eqt. of motion $=$ deformat$^{\text{n}}$ of cpx. str. $\xrightarrow[\text{Calabi conj.}]{\text{Yau.}}$ Ricci flat metrics

- leading cubic vertex of Zwiebach's string action in topo. B-model.
- Barannikov-Kontsevich construction of Frobenius mfd. (via polyvector fields)

<u>Costello-Li 2012</u>. <u>Full</u> string action B-model on CY of arbitrary dim. s.t. BV master eqt. (odd version of Hamilton-Jacobi) $\rightsquigarrow$ BCOV.

<u>Costello-Li 2015, 2016</u> B-model open-closed coupling in large N limit and twisted supergravity.

<u>Convention</u> $\mathbb{Z}$-graded v.s. $V = \bigoplus_{m\in\mathbb{Z}} V_m \ni a$ deg $|a| = m$

deg. shift $V[n]$ w/ $V[n]_m = V_{n+m}$: move left

$$\mathrm{Sym}^m(V) = V^{\otimes m} / a\otimes b - (-1)^{|a||b|} b \otimes a$$

$$\Lambda^m(V) = V^{\otimes m} / a\otimes b + (-1)^{|a||b|} b \otimes a$$

$$\mathrm{Sym}^m(V[1]) = \Lambda^m(V)[m]$$

$$\mathrm{Sym}(V) = \bigoplus_m \mathrm{Sym}^m(V) \qquad \widehat{\mathrm{Sym}}(V) = \prod_m \mathrm{Sym}^m(V)$$

§2 Calabi-Yau Geometry $X^{d\mathbb{C}}$ w/ T_X holo. tangent bdl

$$PV(X) = \bigoplus_{i,j} PV^{i,j}(X) := \Omega^{0,j}(X, \wedge^i T_X)$$

$$\mu \overset{loc.}{=} \sum_{\substack{|I|=i \\ |J|=j}} \mu^I_{\bar J}\, d\bar z^J \otimes \partial_{z^I} \qquad I = \{k_1, \dots, k_i\} \quad J = \{s_1, \dots, s_j\} \qquad \partial_{z^I} = \partial_{z^{k_1}} \wedge \cdots \wedge \partial_{z^{k_i}} \quad d\bar z^J = d\bar z^{s_1} \wedge \cdots \wedge d\bar z^{s_j}$$

$|\mu| = i + j$ total deg.

(wedge) product $PV^{i_1,j_1}(X) \otimes PV^{i_2,j_2}(X) \xrightarrow{\wedge} PV^{i_1+i_2,\, j_1+j_2}(X)$

$(PV(X), \wedge)$ graded comm. alg. $\alpha \wedge \beta = (-1)^{|\alpha||\beta|} \beta \wedge \alpha$

$\bar\partial : PV^{i,j}(X) \longrightarrow PV^{i,j+1}(X) \qquad \bar\partial(\alpha \wedge \beta) = (\bar\partial \alpha) \wedge \beta + (-1)^{|\alpha|} \alpha \wedge \bar\partial \beta$

Def: Calabi-Yau manifold (X, Ω_X) w/ Ω_X nowhere vanishing holo. vol. form

$\Omega_X = e^{\varphi(z)} dz^1 \wedge \cdots \wedge dz^d$ in loc. holo. coord. on X

$\rightsquigarrow$ linear isom. $PV^{i,j} \xrightarrow[\sim]{\lrcorner \Omega_X} \Omega^{d-i,j}$, $\mu \longmapsto \mu \lrcorner \Omega_X$

$\rightsquigarrow$ divergent $\partial_\Omega : PV^{i,j} \longrightarrow PV^{i-1,j}$

$$\begin{tikzcd} PV^{i,j} \arrow[r, "\partial_\Omega"] \arrow[d, "\lrcorner \Omega_X"', "\wr"] & PV^{i-1,j} \arrow[d, "\wr"', "\lrcorner \Omega_X"] \\ \Omega^{d-i,j} \arrow[r, "\partial"] & \Omega^{d-i+1,j} \end{tikzcd}$$

Note: $PV^{i,j} \xrightarrow{\partial_\Omega} PV^{i-1,j}$ NOT a derivation

$\{\alpha, \beta\} := \partial_\Omega(\alpha \wedge \beta) - (\partial_\Omega \alpha) \wedge \beta - (-1)^{|\alpha|} \alpha \wedge \partial_\Omega \beta$ BV alg.

$= -(-1)^{|\alpha|} [\alpha, \beta]_{SN}$ Schouten-Nijenhuis bracket.

Note: $\{-,-\} : PV^{i_1,j_1}(X) \otimes PV^{i_2,j_2}(X) \longrightarrow PV^{i_1+i_2-1,\, j_1+j_2}(X)$

intrinsically def^{d}. (indep. of Ω_X)

From $\{-,-\}$ to ∂_Ω, or Ω_X, $\sim$ BV quantizat^{n}.

Properties: $\{\alpha, \beta\} = (-1)^{|\alpha||\beta|} \{\beta, \alpha\}$

$\{\alpha, \beta \wedge \gamma\} = \{\alpha, \beta\} \wedge \gamma + (-1)^{|\beta||\gamma|} \{\alpha, \gamma\} \wedge \beta$

$\{\{\alpha, \beta\}, \gamma\} = -(-1)^{|\alpha|} \{\alpha, \{\beta, \gamma\}\} + (-1)^{(|\alpha|+1)|\beta|} \{\beta, \{\alpha, \gamma\}\}$

Namely, $(PV, \bar\partial, \partial_\Omega, \{-,-\})$ is dGBV alg.

Trace $\mathrm{Tr} : PV(X) \longrightarrow \mathbb{C}$

$\mathrm{Tr}(\mu) = \int_X (\mu \lrcorner \Omega_X) \wedge \Omega_X$, only non-trivial for $PV^{d,d}$

Ex. $\mathrm{Tr}((\bar\partial \alpha) \beta) = -(-1)^{|\alpha|} \mathrm{Tr}(\alpha (\bar\partial \beta))$

$\mathrm{Tr}((\partial_\Omega \alpha) \beta) = (-1)^{|\alpha|} \mathrm{Tr}(\alpha (\partial_\Omega \beta))$

Deformation theory and local moduli

X cpt Kähler $\quad \mathcal{M}^{cx} \overset{\text{locally}}{=} \{\mu \in PV^{1,1}(X)_{\|\cdot\|<\varepsilon} \mid \bar\partial\mu + \frac{1}{2}\{\mu,\mu\} = 0,\ \overset{\text{gauge fixing}}{\bar\partial^*\mu = 0}\}$

[Bogomolov-Tian-Todorov]

X cpt. Kähler CY $\Longrightarrow \mathcal{M}^{cx}$ smooth, tangent sp. $H^1(X, T_X)$.

Extended moduli space $\quad \mathcal{M} \overset{\text{loc.}}{=} \{\bar\partial\mu + \frac{1}{2}\{\mu,\mu\} = 0,\ \bar\partial^*\mu = 0\}$, not nec. type $(1,1)$

[Barannikov-Kontsevich] $\mathcal{M}$ smooth

Pf. $\alpha, \beta \in \operatorname{Ker}\partial_\Omega \subset PV$

$$\{\alpha,\beta\} = \partial_\Omega(\alpha\wedge\beta) - \cancel{(\partial_\Omega\alpha)}\wedge\beta - (-1)^{|\alpha|}\alpha\wedge\cancel{\partial_\Omega\beta} \in \operatorname{Im}\partial_\Omega \subset \operatorname{Ker}\partial_\Omega$$

$$\underset{\text{dgla}}{(\operatorname{Ker}\partial, \bar\partial, \{-,-\})} \xhookrightarrow{\ j\ } \underset{\text{(shifted) dgla}}{(PV, \bar\partial, \{-,-\})}$$

$$PV(X) \overset{\text{harmonic}}{\Longrightarrow} \mathbb{H} \simeq \operatorname{Ker}\partial_\Omega/\operatorname{Im}\partial_\Omega \simeq \operatorname{Ker}\bar\partial/\operatorname{Im}\bar\partial \quad (\because \text{Kähler})$$

$$\bar\partial(\operatorname{Ker}\partial_\Omega) \subset \operatorname{Im}\partial_\Omega \qquad (\partial\bar\partial\text{-lemma})$$

$$(\operatorname{Ker}\partial, \bar\partial, \{-,-\}) \xrightarrow[\text{harmonic proj.}]{\pi} (\mathbb{H}, 0, 0) \qquad \text{dgla homo.}$$

$$\begin{tikzcd} & (\operatorname{Ker}\partial, \bar\partial, \{-,-\}) \arrow[dl, "j"', "\text{q.isom.}"] \arrow[dr, "\pi", "\text{q.isom.}"'] & \\ (PV, \bar\partial, \{-,-\}) & & (\mathbb{H}, 0, 0) \end{tikzcd}$$

§3 Period map

(X, Ω_X) cpt. Kähler CY

$\mu \in PV^{1,1}(X)$ s.t. $\bar\partial\mu + \frac{1}{2}\{\mu,\mu\} = 0 \quad \rightsquigarrow \quad [X_\mu] \in \mathcal{M}^{cx}_X$ local moduli of cpx str. on X

$\mu = \sum_{i,j} \mu^i_{\bar j}\, d\bar z^j \otimes \partial_{z^i} \quad \Longrightarrow \quad dz^i + \mu \lrcorner dz^i = dz^i + \sum_j \mu^i_{\bar j}\, d\bar z^j = e^\mu \lrcorner dz^i$

wrt $\{z^i\}$ loc. holo. coord. on X $\qquad$ loc. basis of $(1,0)$-forms in X_μ

$$\begin{array}{ccc} \mathcal{H}^{d=\dim_{\mathbb{C}}X} & \supset & H^d(X_\mu) \\ \downarrow & & \downarrow \\ \mathcal{M}^{cx}_X & \ni & [X_\mu] \end{array}$$

$\exists$ Gauss-Manin (flat) connection ∇^{GM}

$\rightsquigarrow$ identify $H^d(X_\mu) \simeq H^d(X) = H^d(X,\mathbb{C})$ loc. on $\mathcal{M}^{cx}_X$

$\mathcal{H}^d \supset F^p\mathcal{H}^d$ holom. subbdl. w/ fiber $\bigoplus_{k\geq p} H^{k,d-k}(X_\mu)$

In particular, $L = F^n\mathcal{H}^d$ holo. line bdl., called vacuum line bdl.

period map

$$\begin{array}{ccc} \mathcal{M}^{cx}_X & \xhookrightarrow{\text{loc. Torelli}} & \mathbb{P}(H^d(X,\mathbb{C})) \\ {[X_\mu]} & \longmapsto & [\Omega_{X_\mu}] \end{array}$$

$$\begin{tikzcd} \operatorname{Tot}(L\setminus\{0\}) = \hat{\mathcal{M}}^{cx} \arrow[r, hook] \arrow[d, "\mathbb{C}^*\text{-bdl}/\mathcal{M}^{cx}_X"'] & H^d(X,\mathbb{C}) \\ \mathcal{M}^{cx} & \end{tikzcd}$$

$[X_\mu, \Omega_{X_\mu}] \longmapsto [\Omega_{X_\mu}]$

CY3 $(X,\Omega_X) \rightsquigarrow H^3(X,\mathbb{C})$ w/ cx. sympl. form $\omega(\alpha,\beta)=\int_X \alpha\wedge\beta$

$F^2H^3 = H^{3,0}\oplus H^{2,1} \subset H^3$ Lagr. subsp.

$\rightsquigarrow$ Lagr. subbdl.
$$\begin{tikzcd} F^2\mathcal{H}^3 \arrow[r, phantom, "\subset"] \arrow[d] & \mathcal{H}^3, \omega \arrow[d] \\ \mathcal{M}^{cx} \arrow[r, equal] & \mathcal{M}^{cx} \end{tikzcd}$$
sympl. vector bdl. $\nabla^{GM}\omega = 0$

Prop. $\hat{\mathcal{M}}^{cx} \hookrightarrow H^3(X,\mathbb{C})$ Lagr. submfd.
(reason: Griffiths transversality)

Choose linear Lagr. $\mathcal{L}\subset H^3(X,\mathbb{C})$, transverse to F^2H^3

$\Longrightarrow$ $H^3 = T^*(F^2H^3)$ as sympl. v.s.

$\overset{\text{Lagr.}}{\uparrow}$

$\hat{\mathcal{M}}^{cx} \Longrightarrow \hat{\mathcal{M}}^{cx} = \mathrm{Graph}(d\tilde{\mathcal{F}}_0)$ $\exists$ holo. $\hat{\mathcal{F}}_0 : \overbrace{F^2H^3}^{H^{3,0}\oplus H^{2,1}} \longrightarrow \mathbb{C}$

Eg. $\mathcal{L} = H^{1,2}\oplus H^{0,3}$ cpx. conjugate splitting.

$\rightsquigarrow$ $\hat{\mathcal{M}}^{cx} = \{(t^0, t^i, \partial_{t^i}\hat{\mathcal{F}}_0, \partial_{t^0}\hat{\mathcal{F}}_0)\}$

$\hat{\mathcal{F}}_0$ has homog. deg. 2 wrt $\mathbb{C}^\times \curvearrowright \hat{\mathcal{M}}^{cx}$, $\hat{\mathcal{M}}^{cx}/\mathbb{C}^\times = \mathcal{M}^{cx}$

$\Longrightarrow$ $\hat{\mathcal{F}}_0 = (t^0)^2\mathcal{F}_0(\tau^i)$ where $\tau^i := t^i/t^0$ loc. coord. on $\mathcal{M}^{cx}$

Geometrically, $\mathcal{F}_0 \in \Gamma(\mathcal{M}^{cx}, L^{\otimes(-2)})$, called prepotential

$\rightsquigarrow$ $\hat{\mathcal{M}}^{cx} = \{t^0(1,\tau^i,\partial_{\tau^i}\mathcal{F}_0, 2\mathcal{F}_0 - \tau^i\partial_{\tau^i}\mathcal{F}_0)\}$

i.e. $[\Omega_\tau] = (1,\tau^i, \partial_{\tau^i}\mathcal{F}_0, 2\mathcal{F}_0-\tau^i\partial_{\tau^i}\mathcal{F}_0)$

Yukawa coupling $C_{ijk}(\tau) = \int_X \nabla^{GM}_{\tau^i}[\Omega_\tau]\wedge \nabla^{GM}_{\tau^j}\nabla^{GM}_{\tau^k}[\Omega_\tau] \overset{Ex}{=} \partial_{\tau^i}\partial_{\tau^j}\partial_{\tau^k}\mathcal{F}_0(\tau)$

Deformation theory for pairs (X,Ω_X), any dim d

deform X ~ $\mu\in PV^{1,1}(X)$, $\bar\partial\mu + \frac{1}{2}\{\mu,\mu\}=0$

$\rightsquigarrow$ X_μ w/ holo. (1,0)-form $e^\mu \lrcorner dz^i$'s

$\rightsquigarrow$ non-vanishing (d,0)-form $e^\mu\lrcorner\Omega_X$ $(\because e^\mu\lrcorner(\alpha\wedge\beta) = (e^\mu\lrcorner\alpha)\wedge(e^\mu\lrcorner\beta))$

$\rightsquigarrow$ $\exists$smooth fu. ρ, $d(e^\rho e^\mu\lrcorner\Omega_X)=0$ $\rightsquigarrow$ holom. vol. form

$\Longleftrightarrow$ $\bar\partial\mu+\frac{1}{2}\{\mu,\mu\}=0$ & $\bar\partial\rho + \partial_\Omega\mu + \{\mu,\rho\}=0$ $\quad(*)$

§4 Extended period map

$Q = \bar\partial + t\,\partial_\Omega$ w/ t formal variables, $\deg t = 2$ ($\sim$gravitational descendant)

(*) $\iff$ $Q\hat\mu + \frac{1}{2}\{\hat\mu,\hat\mu\} = 0$ w/ $\hat\mu = \mu + t\rho$ $(\mu \in PV^{1,1}, \rho \in PV^{0,0})$

$(PV, \bar\partial, \{-,-\}) \rightsquigarrow$ extended moduli of X

$(PV[\![t]\!], Q, \{-,-\}) \rightsquigarrow$ extended moduli of CY geometry

Remark: $(PV[\![t]\!], Q)$ homological replacement of $(\mathrm{Ker}\,\partial, \bar\partial)$.

$$\mathrm{Ker}\,\partial \longrightarrow [PV \xrightarrow{t\partial} tPV \xrightarrow{t\partial} t^2PV \xrightarrow{t\partial} \cdots]$$

Question: (*) as Euler-Lagrange eqt. ?

<u>BCOV</u> on CY3 for $\mu \in \mathrm{Ker}\,\partial$ $\leftarrow$ not local !

$\bar\partial\mu + \frac{1}{2}\{\mu,\mu\} = 0$ as EL-eqt. of KS Kodaira-Spencer action

$$Q\hat\mu + \frac{1}{2}\{\hat\mu,\hat\mu\} = 0 \qquad (\text{not EL-eqt})$$

$\wr\!\downarrow$ $\exists$ L_∞-transf. of dgla (Costello-Li) uses period map

$$Q\hat\mu + \sum_{k=2}^{\infty} \frac{1}{k!} l_k(\mu^{\otimes k}) = 0 \qquad \text{EL-eqt}, \exists \text{ action fcl.}$$

Def. $S(X) = S_+(X) \oplus S_-(X)$ as

$$PV(X)(\!(t)\!)[2] = PV(X)[\![t]\!][2] \oplus t^{-1}PV(X)[t^{-1}][2]$$

Note: $(S_+(X), Q) \subset (S(X), Q)$ sub-cpx.

Def: $\omega : S(X) \otimes S(X) \longrightarrow \mathbb{C}$

$$\omega(f(t)\alpha, g(t)\beta) := \mathrm{Tr}\,\alpha\beta \cdot \mathrm{Res}_{t=0} f(t)g(-t)\,dt$$

$$= -(-1)^{|\alpha||\beta|}\,\omega(g(t)\beta, f(t)\alpha)$$

Only nontrivial if total deg. is $2d-2$

$\rightsquigarrow$ $(S(X), \omega)$ is $(6-2d)$-shifted sympl. v.s.

Eg CY3

$$\begin{array}{ccccccc} \Omega^{3,0} & \oplus & \Omega^{2,1} & \oplus & \Omega^{1,2} & \oplus & \Omega^{0,3} \\ & & & \text{period} \updownarrow \lrcorner\Omega_X & & & \\ tPV^{00} & \oplus & PV^{1,1} & \oplus & t^{-1}PV^{22} & \oplus & t^{-2}PV^{33} \end{array}$$

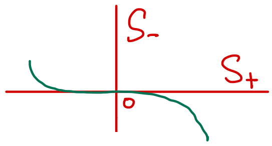
Q (graded) skew-symm. w.r.t. ω.

Cochain period map: (a map on formal schemes)

$$\mathcal{P} : S_+(X) \longrightarrow S(X) \qquad \mathcal{P}(\mu) = t(1 - e^{\mu/t})$$

$$d\mathcal{P}(0) : T_0 S_+(X) \longrightarrow T_0 S(X)$$

is $S_+(X) \xhookrightarrow{(-1)} S(X)$



(In particular, $\mathcal{P}$ is a formal embedding)

Ex. $Q(t(1 - e^{\mu/t})) = -(Q\mu + \frac{1}{2}\{\mu, \mu\})e^{\mu/t}$

That is $\mathcal{P} : (S_+(X), Q, \{-,-\}) \longrightarrow (S(X), Q, 0)$

If we view Q as a vector field on $S(X)$, then Q is tangent to $\mathrm{Im}(\mathcal{P})$.

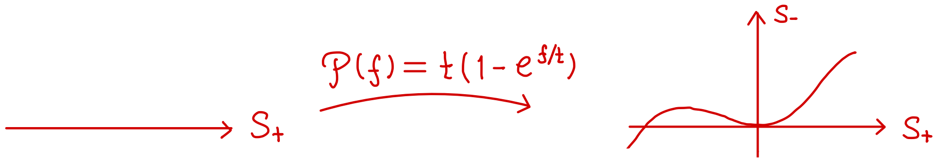
$\rightarrow$ $\mathrm{Im}(\mathcal{P}) \subset S(X)$ is a formal shifted Lagr. submfd.

§5 Classical BCOV Equation

Geometry of gravitational descendant. Consider $X = pt$.

$$S = \mathbb{C}((t)) \supset S_+ = \mathbb{C}[[t]], \quad S_- = t^{-1}\mathbb{C}[t^{-1}]$$

$$\omega(f(t), g(t)) = \mathrm{Res}_{t=0} f(t)\, g(-t)\, dt \qquad \text{sympl. form}$$



$\mathrm{Im}(p) \subset S$ is a formal Lagr. submanifold.

$$S \underset{\text{sum}}{\overset{\text{Lagr.}}{=\!=}} S_+ \oplus S_- \Longrightarrow S = T^*S_+ \supset \mathrm{Im}\,\mathcal{P} = \mathrm{Graph}(d I_0) \quad \exists\, I_0 \in \mathcal{O}(S_+)$$

$$\text{Write} \quad \tau = \sum_{k=0}^{\infty} \tau_k t^k \in S_+$$

$$\Rightarrow \mathcal{P}(\tau) = \sum_{k=0}^{\infty} \tau_k t^k + \sum_{k=0}^{\infty} \partial_{\tau_k} I_0 \, (-t)^{-(k+1)}$$

Goal: Compute $I_0(\tau)$

- <u>String equation</u> Let $\hat{t} : S \longrightarrow S$, $\hat{t}(t^k) = t^{k+1}$

View $\hat{t}^{-1}$ as a vector field on S, denote L_{-1}.

$$L_{-1} \text{ preserves } \omega \quad \text{i.e.} \quad \omega(L_{-1}f, g) = -\omega(f, L_{-1}g)$$

$$L_{-1} = \sum_i \tau_{i+1} \frac{\partial}{\partial \tau_i} \quad \& \quad \omega = \sum_{k\geq 0}^{\infty} (-1)^{k+1} d\tau_k \wedge d\tau_{-k-1} = \frac{1}{2} \sum_{i+j=-1}^{\infty} (-1)^j d\tau_i \wedge d\tau_j$$

$$\Longrightarrow \iota_{L_{-1}} \omega = \frac{1}{2} \sum_{i+j=-1}^{\infty} (-1)^j \tau_{i+1} \wedge d\tau_j = \frac{1}{2}\left(\sum_{i+j=0}^{\infty} (-1)^j \tau_i \tau_j \right)$$

$$= d\left(\frac{1}{2}\tau_0^2 + \sum_{k\geq 1} (-1)^k \tau_k \tau_{-k} \right)$$

$$\rightsquigarrow \text{Hamiltonian } h_{-1} = \frac{1}{2}\tau_0^2 + \sum_{k\geq 1} (-1)^k \tau_k \tau_{-k}$$

Lemma: L_{-1} is tangent to $\mathrm{Im}\,\mathcal{P} - t$ (dilaton shift)

Pf: Let $\alpha = \mathcal{P}(f) - t = -t e^{f/t}$

$$T_\alpha(\mathrm{Im}\,\mathcal{P} - t) = \{-\delta f\, e^{f/t}\} = e^{f/t} S_+$$

$$L_{-1}|_\alpha = L_{-1}(\alpha) = -e^{f/t} \subset T_\alpha(\mathrm{Im}\,\mathcal{P} - t) \qquad \#$$

By Hamilton-Jacobi eqt. $h_{-1}|_{\mathrm{Im}\mathcal{P}-t} = \text{Const.} = 0$

$$\Longrightarrow \frac{1}{2}\tau_0^2 + \sum_{k=0}^{\infty} \tau_{k+1} \partial_{\tau_k} I_0 - \partial_{\tau_0} I_0 = 0$$

$$\text{i.e.} \quad \partial_{\tau_0} I_0 = \frac{1}{2}\tau_0^2 + \sum_{k=0}^{\infty} \tau_{k+1} \frac{\partial}{\partial \tau_k} I_0 \qquad \text{String eqt.}$$

- <u>Dilaton eqt.</u> $L_0 : S \longrightarrow S$, $L_0(t^k) = (k+\frac{1}{2})t^k$ s.t. $\omega(L_0 f, g) = -\omega(f, L_0 g)$

  $\rightsquigarrow$ Hamiltonian $h_0 = \sum_{k\geq} (k+\frac{1}{2})(-1)^{k-1} \tau_k \tau_{-k-1}$

  Lemma: L_0 is tangent to $\mathrm{Im}\mathcal{P} - t$ (similar pf.)

  By Hamilton-Jacobi eqt. $h_0|_{\mathrm{Im}\mathcal{P}-t} = \text{Const.} = 0$

  $$\Longrightarrow \frac{3}{2}\frac{\partial}{\partial \tau_1} I_0 = \sum_{k=0}^{\infty} (k+\tfrac{1}{2})\tau_k \frac{\partial}{\partial \tau_k} I_0 \qquad \text{Dilaton eqt.}$$

- <u>Grading eqt.</u> $G : S \longrightarrow S$, $G(t^k) = (2k-2)t^k$

  Lemma: G is tangent to $\mathrm{Im}\mathcal{P}$

  i.e. $\mathrm{Im}\mathcal{P}$ is preserved by $\mathbb{C}^*$-action $\lambda \cdot t^k = \lambda^{2k-2} t^k$.

  $\rightsquigarrow$ grading on $\mathbb{C}((t))[2]$ $\deg t^k = 2k-2$ $\deg \tau_k = 2-2k$

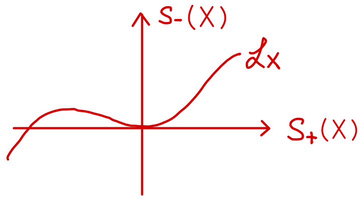
  $\rightsquigarrow$ I_0 homog. of degree 6

  string eqt., dilaton eqt., grading $\Rightarrow$

  $$\frac{\partial}{\partial \tau_{k_1}} \cdots \frac{\partial}{\partial \tau_{k_n}}\Big|_{\tau=0} I_0 = \binom{n-3}{k_1 k_2 \cdots k_n} = \int_{\overline{\mathcal{M}}_{0,n}} \psi_1^{k_1} \cdots \psi^{k_n}$$

- <u>Classical BCOV action</u> X

  $$S_+(X) = PV(X)[[t]][2] \qquad \mathcal{P}(f) = t(1-e^{f/t})$$



$$S(X) = S_+(X) \oplus S_-(X) = T^* S_+(X)$$

$$\Longrightarrow \quad \mathcal{L}_X := \mathrm{Im}\mathcal{P} = \mathrm{Graph}(dI_0^X) \quad \exists\, I_0^X \in \mathcal{O}(S_+(X))$$

Denote $\langle t^{k_1} \otimes \cdots \otimes t^{k_n} \rangle_0 := \int_{\overline{\mathcal{M}}_{0,n}} \psi_1^{k_1} \cdots \psi^{k_n} = \binom{n-3}{k_1 k_2 \cdots k_n}$

which extends $PV(X)$-linearly to

$$\langle - \rangle_0 : \mathrm{Sym}^{\bullet}(S_+(X)) \longrightarrow PV(X)$$

Theorem (Costello-Li) $I_0^X(\mu) = \mathrm{Tr}\langle e^{\mu} \rangle_0 = \sum_{n\geq 3} \frac{1}{n!} \mathrm{Tr}\langle \mu^{\otimes n} \rangle_0$ classical BCOV action

If we write $\mu = \mu_0 + \mu_1 t + \mu_2 t^2 + \cdots$ w/ $\mu_k \in PV(X)$

$$I_0^X(\mu) = \frac{1}{3!} \mathrm{Tr}\, \mu_0^3 \quad (\mathrm{mod}\ \mu_{>0})$$

For <u>CY3</u>, BCOV's Kodaira-Spencer gravity,

fields $\mu \in \mathrm{Ker}\, \partial \subset PV(X)$ (non-local)

action $KS[\mu_0] = \frac{1}{2} \mathrm{Tr}\, \mu_0 \frac{\bar{\partial}}{\partial} \mu_0 + \frac{1}{6} \mathrm{Tr}\, \mu^3$

§6 Hodge structure and primitive forms

$(S_+(X), Q=\bar\partial + t\partial, \{-,-\})$ & $\mathcal{P}$ $\longrightarrow$ pass to cohomology (Maurer-Cartan functor)

$\rightsquigarrow$ $\mathcal{M}_X = \{\mu \in S_+(X) \mid Q\mu + \frac{1}{2}\{\mu,\mu\} = 0\} / \text{Gauge}$

$\mathcal{P} : \mathcal{M}_X \hookrightarrow H^\bullet(S(X), Q)$ $(6-2d)$-shifted sympl. ($\because$ Q is compati. w/ ω)

extended period map. $\quad \cup$ isotropic subsp.

$H^\bullet(S_+(X), Q)$ $\quad$ ($\because$ $(S_-(X), Q) \subset (S(X), Q)$ sub-cpx.)

but $S_+(X) \subset S(X)$ not preserve by Q.

$\Longrightarrow \nexists\ \cancel{H^\bullet(S_-(X), Q)}$

Def. An opposite filtration of $H^\bullet(S(X), Q)$ is a linear isotropic subspace

$$\mathcal{L} \subset H^\bullet(S(X), Q) \text{ s.t. } H^\bullet(S(X), Q) = H^\bullet(S_+(X), Q) \oplus \mathcal{L} \text{ and } t^{-1}\mathcal{L} \subset \mathcal{L}.$$

$$\rightsquigarrow \begin{tikzcd} \mathcal{L} \arrow[r, hook] \arrow[rrd, bend right] & H^\bullet(S(X), Q) \arrow[r, two heads] & H^\bullet(S_+(X), Q)/tH^\bullet(S_+(X), Q) \arrow[d, "\wr"] \\ & & H^\bullet(PV(X), \bar\partial) \end{tikzcd}$$

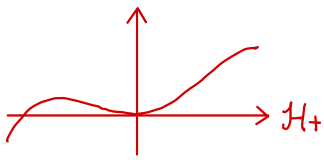
$$\rightsquigarrow \quad B^{\mathcal{L}} := H^\bullet(S_+(X), Q) \cap t\mathcal{L} \xrightarrow{\simeq} H^\bullet(PV(X), \bar\partial)$$

denote $\mathcal{H} := H^\bullet(S(X), Q) = B^{\mathcal{L}}((t))$

$\mathcal{H}_+ := H^\bullet(S_+(X), Q) = B^{\mathcal{L}}[[t]]$

$\longrightarrow$ $\mathcal{L} = t^{-1}B^{\mathcal{L}}[t]$; $\quad t^k B^{\mathcal{L}}$ & $t^{-(k+1)}B^{\mathcal{L}}$ are ω-conjugate to each other.

$$\mathcal{M}_X \xrightarrow{\mathcal{P}} \mathcal{H} \overset{\text{via } \mathcal{L}}{=\!=} T^*\mathcal{H}_+$$



$$\mathcal{P}(\mathcal{M}_X) = \text{Graph}(d\mathcal{F}_0^{X,\mathcal{L}})$$

$\exists$ fu. $\mathcal{F}_0^{X,\mathcal{L}}$ on $\mathcal{H}_+ = H^\bullet(PV(X), \bar\partial)[[t]]$, $g=0$ B-model invariants.

Geometry of the opposite filtration $\mathcal{L}$

Define $\Gamma_\Omega : S(X) = PV(X)((t)) \longrightarrow \Omega(X)((t))$

$$\Gamma_\Omega(t^k \mu) = t^{k+i-1} \mu \lrcorner \Omega \qquad \mu \in PV^{i,j}$$

$$Q = \bar\partial + t\partial \longleftrightarrow d = \bar\partial + \partial$$

$$\Gamma_\Omega(S_+(X)) = \prod_{p \in \mathbb{Z}} t^{d-p-1} F^p\Omega(X)$$

where $F^p\Omega(X) = \Omega^{\geq p,*}(X)$, the Hodge filtration

$$\Longrightarrow \quad \Gamma_\Omega(t^k PV^{i,*}) = t^{k+i-1}\Omega^{d-i,*}(X) \overset{p=d-i-k}{=\!=\!=} t^{d-p-1}\Omega^{p+k,*}(X)$$

In particular, $\Gamma_\Omega : S_+(X)/t\,S_+(X) \xrightarrow{\sim} \prod_p t^{d-p-1} Gr^p\, \Omega(X)$

$\bar\partial \longleftrightarrow \bar\partial$

$\rightsquigarrow$ $\mathcal{L}$ is a spliting of Hodge filtr$^{\underline{n}}$.

Eg. 1. $\mathcal{L} = \overline{\mathcal{F}^p}$ Harmonic splitting $\rightsquigarrow$ $H^n(X) = \bigoplus_{p+q=n} H^{p,q}(X)$

Eg. 2. $\mathcal{L}$ = monodromy splitting arise from LCSL

<u>CY3</u>

$$\begin{array}{c} tPV^{0,0} \oplus PV^{1,1} \oplus t^{-1} PV^{2,2} \oplus t^{-2} PV^{3,3} \underset{\text{sympl. subsp.}}{\subset} PV(X)((t)) \\ \Gamma_\Omega \downarrow \cong \\ \Omega^{3,0} \oplus \Omega^{2,1} \oplus \Omega^{1,2} \oplus \Omega^{0,3} \end{array}$$

• Primitive form (K. Saito) / J-function (Givental)

Fix splitting $\mathcal{L} \rightsquigarrow B^\alpha := H^\bullet(S_+(X), Q) \cap \mathcal{L} \xrightarrow{\simeq} H^\bullet(PV(X), \bar\partial)$

choose basis φ_α's

Def$^{\underline{n}}$. $K : \mathcal{H} \otimes \mathcal{H} \longrightarrow \mathbb{C}((t))$ Higher residue pairing

$K(\alpha f(t), \beta g(t)) = Tr(\alpha\beta) \cdot f(t) g(t)$

$\omega(\mu, \nu) = \oint K(\mu, \nu)\, dt$

modulo t, $K = Tr$ $\quad H^\bullet(PV(X), \bar\partial) = H^\bullet(X, \wedge^\bullet T_X)$

$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{L}$: ω-isotropic splitting

$K(\mathcal{H}_+, \mathcal{H}_+) = 0$, $\quad K(\mathcal{L}, \mathcal{L}) \subset t^{-2}\, \mathbb{C}[t^{-1}]$

$K : B^\alpha \otimes B^\alpha \longrightarrow \mathbb{C}$ where $B^\alpha = \mathcal{H}_+ \cap t\mathcal{L}$.

$K(\varphi_\alpha, \varphi_\beta) = \eta_{\alpha\beta}$ w/o higher t, such is called a 'good basis'.

define $\mathcal{M}_X^\circ$:

$$\begin{tikzcd} \mathcal{M}_X^\circ \arrow[r, hook] \arrow[d, "\cong"'] \arrow[dr, phantom, "\square"] & \mathcal{M}_X \arrow[d, "\cong", "\Pi_X^\alpha \circ \mathcal{P}"'] \\ B^\alpha \arrow[r, hook] & \mathcal{H}_+ \end{tikzcd}$$

$\{\tau_0^\alpha, \tau_{\geq 1}^\alpha = 0\}$ $\quad \{\tau_k^\alpha\}_{k \geq 0}$

$\left(\Pi_+^\alpha : \underbrace{\mathcal{H}}_{B^\alpha((t))} = \mathcal{H}_+ \oplus \mathcal{L} \xrightarrow{\text{proj.}} \underbrace{\mathcal{H}_+}_{B^\alpha[[t]]} \right)$

$\rightsquigarrow$ family $\mu^\alpha(\tau_0^\alpha) \in PV(X)[[t]]$

s.t. $Q\mu^\alpha(\tau_0^\alpha) + \frac{1}{2}\{\mu^\alpha(\tau_0^\alpha), \mu^\alpha(\tau_0^\alpha)\} = 0$

$\Pi_+^\alpha\, [t(1 - e^{\mu^\alpha/t})]_Q = \sum_\alpha \tau_0^\alpha \varphi_\alpha$

means Q-cohom. class.

$$[t(1-e^{\mu^\alpha/t})]_Q = \sum_\alpha \tau_0^\alpha \varphi_\alpha + \sum_{k=0}^{\infty} (-t)^{-k-1} \eta^{\alpha\beta} \frac{\partial \mathcal{F}_0^\alpha}{\partial \tau_k^\beta} \varphi_\beta \Big|_{\tau_{>0}} =: J(\tau_0^\alpha, t) \quad \text{J-function}$$

$$[e^{\mu^\alpha/t}]_Q = 1 - t^{-1} J \qquad \text{primitive form}$$

Prop: $K(\partial_{\tau_0^\alpha} J, \partial_{\tau_0^\beta} J) = \eta_{\alpha\beta}$ (indep. of t)

Pf. $K(\partial_{\tau_0^\alpha} J, \partial_{\tau_0^\beta} J) = K(\varphi_\alpha + O(t^{-1}), \varphi_\beta + O(t^{-1})) = \eta_{\alpha\beta} + O(t^{-1})$

$= K(\partial_{\tau_0^\alpha}[t(1-e^{\mu^\alpha/t})], \partial_{\tau_0^\beta}[t(1-e^{\mu^\alpha/t})])$

$= K((\partial_{\tau_0^\alpha}\mu^\alpha) e^{\mu^\alpha/t}, (\partial_{\tau_0^\beta}\mu^\alpha) e^{\mu^\alpha/t}) \in \mathbb{C}[\![t]\!] \longrightarrow \eta_{\alpha\beta}$

$(\because \mu^k = \mu_0 + t\mu_1 + \ldots)$

$\Longrightarrow$ $\partial_{\tau_0^\alpha} J$'s defines a family of 'good basis' param. by $\{\tau_0^\alpha\}$

Geometrically, $\mathrm{Span}\{\partial_{\tau_0^\alpha} J\}_{\tau_0^\alpha} = T_{\tau_0^\alpha}\mathcal{M}_X \cap t\mathcal{L}$

At $\tau_0^\alpha = 0$, $T_0\mathcal{M}_X \cap t\mathcal{L} = \mathcal{H}_+ \cap t\mathcal{L} = B^\alpha$

$\mathcal{H} = T_0\mathcal{M}_X \oplus \mathcal{L} \Longrightarrow \mathcal{H} = T_{\tau_0^\alpha}\mathcal{M}_X \oplus \mathcal{L}$ for nearby τ_0^α

$\{\partial_{\tau_0^\alpha} J\}$ forms an basis of $T_{\tau_0^\alpha}\mathcal{M}_X \cap t\mathcal{L}$

Compute $\partial_{\tau_0^\alpha}\partial_{\tau_0^\beta} J$: (1) $\partial_{\tau_0^\alpha}\partial_{\tau_0^\beta} J \in \mathcal{L}$ by the expression of J

(2) $\partial_{\tau_0^\alpha}\partial_{\tau_0^\beta} J = \partial_{\tau_0^\alpha}\partial_{\tau_0^\beta}[t - te^{\mu^\alpha/t}]_Q = -[(\frac{1}{t}\partial_{\tau_0^\alpha}\mu^\alpha \partial_{\tau_0^\beta}\mu^\alpha + \partial_{\tau_0^\alpha}\partial_{\tau_0^\beta}\mu^\alpha) e^{\mu^\alpha/t}]_Q \in \frac{1}{t} T_{\tau_0}\mathcal{M}_X$

$\Longrightarrow \partial_{\tau_0^\alpha}\partial_{\tau_0^\beta} J \in \frac{1}{t} T_{\tau_0}\mathcal{M}_X \cap \mathcal{L} = \frac{1}{t}(T_{\tau_0}\mathcal{M}_X \cap t\mathcal{L}) = \frac{1}{t}\mathrm{Span}\{\partial_{\tau_0^\alpha} J\}$

$\Longrightarrow \exists\, A_{\alpha\beta}^\gamma(\tau_0^\alpha)$ s.t. $\partial_{\tau_0^\alpha}\partial_{\tau_0^\beta} J + \frac{1}{t} A_{\alpha\beta}^\gamma(\tau_0)\, \partial_{\tau_0^\gamma} J = 0$ Quantum Differential Eqt.

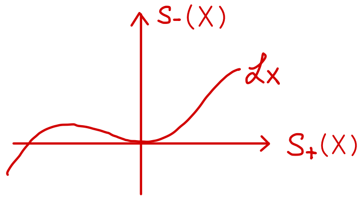
solve WDVV eqt.

Equivalently, $(\partial_{\tau_0^\alpha}\partial_{\tau_0^\beta} + \frac{1}{t} A_{\alpha\beta}^\gamma(\tau_0)\, \partial_{\tau_0^\gamma})[e^{\mu^\alpha/t}] = 0$

Remark: Perturbative algorithm for LG model by Li-Li-Saito.

§7 Classical BV master equation

$$S_+(X) = PV(X)[[t]][2] \xrightarrow{\mathcal{P}(f) = t(1-e^{\mu/t})}$$



$$\mathcal{L}_X := \mathrm{Im}\,\mathcal{P} = \mathrm{Graph}(d I_0^X) \quad \text{w/} \quad I_0^X(\mu) = \mathrm{Tr}\langle e^{\mu}\rangle_0$$

Q preserves ω $\Longrightarrow$ quadratic Hamiltonian $h_Q|_{\mathcal{L}_X} = 0$

$$Q = \bar\partial + t\,\partial \cdot \ S(X) \longrightarrow S(X), \qquad Q^2 = 0 \quad \text{cohomological v.f.}$$

$$Q : \begin{cases} S_+(X) \longrightarrow S_+(X) \\ S_-(X) \longrightarrow S_+(X) \oplus S_-(X) \end{cases}$$

<u>Toy model</u> (finite dim.)

Let (V, ω, Q) dg sympl. v.s.

ω 0-sympl. $\quad \omega \in \Lambda^2 V^*$, $\quad \omega^{-1} \in \Lambda^2 V$

Q skew-symm. w.r.t. ω.

Assume Lagrangian splitting $V = V_+ \oplus V_-$

$$Q : \begin{cases} V_+ \longrightarrow V_+ \\ V_- \longrightarrow V_+ \oplus V_- \end{cases} \quad \rightsquigarrow \quad V = T^* V_+ \xrightarrow{\pi_+} V_+$$

$$\omega^{-1} \in (V_+ \otimes V_-) \oplus (V_- \otimes V_+) \subset \Lambda^2 V$$

$$(Q \otimes 1)\,\omega^{-1} \in (V_+ \otimes V_-) \oplus (V_- \otimes V_+) \oplus (V_+ \otimes V_+)$$

$$P := (\pi_+ \otimes \pi_+)\big((Q \otimes 1)\,\omega^{-1}\big) \in V_+ \otimes V_+$$

P measure the failure of Q preserving V_-.

Prop. $P \in \mathrm{Sym}^2(V_+)$ of deg 1 & Q-compatible ($(Q \otimes 1 + 1 \otimes Q)P = 0$)

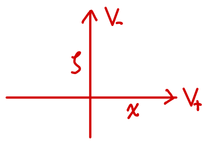
(Pf: $(Q \otimes 1 + 1 \otimes Q)\,\omega^{-1} = 0 \Longrightarrow (Q \otimes 1)\,\omega^{-1} \in \mathrm{Sym}^2(V) \Longrightarrow P \in \mathrm{Sym}^2(V_+)$

Geometrically,

$$\begin{array}{lccc} & (Q \otimes 1)\,\omega^{-1} & \in & \mathrm{Sym}^2(V) \\ & \big\downarrow \iota_\omega & & \simeq \big\downarrow \iota_\omega \\ \text{quad. Hamil. of } Q : & h_Q & \in & \mathrm{Sym}^2(V^*) \end{array}$$

$$(Q \otimes 1)\,\omega^{-1} \in (V_+ \otimes V_-) \oplus (V_- \otimes V_+) \oplus \mathrm{Sym}^2(V_+)$$

$$\Longrightarrow \quad h_Q \in \underset{x\ \ \xi}{(V_+^* \otimes V_-^*)} \oplus (V_-^* \otimes V_+^*) \oplus \underset{\xi\xi}{\mathrm{Sym}^2(V_-^*)}$$



Denote $\partial_P : \mathrm{Sym}^n(V_+^*) \longrightarrow \mathrm{Sym}^{n-2}(V_+^*)$, contracting w/ $P \in \mathrm{Sym}^2(V_+^*)$

Define BV bracket $\{f, g\}_P := \partial_P(fg) - (\partial_P f)g - (-1)^{|f|} f \partial_P g$

Prop. $(\mathcal{O}(V_+) = \mathrm{Sym}^n(V_+^*), Q, \partial_P, \{-,-\}_P)$ is a dGBV-alg.

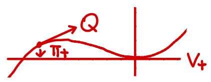
[Pf: $(Q \otimes 1 + 1 \otimes Q) P = 0 \Longrightarrow [Q, \partial_P] = 0$

Given Lagrangian $\mathcal{L} = \mathrm{Graph}(dI_0) \subset T^*V_+ = V$

Q tangent to $\mathcal{L}$

$\Longleftrightarrow$ $Q I_0 + \frac{1}{2}\{I_0, I_0\}_P = 0$ (reason: (odd) Hamilton-Jacobi $h_Q|_{\mathcal{L}} = 0$.)
(classical master eqt.)

$\Longleftrightarrow$ $\delta := Q + \{I_0, -\}_P : \mathcal{O}(V_+) \longrightarrow \mathcal{O}(V_+)$ s.t. $\delta^2 = 0$.

$\delta = (\pi_+)_*[Q|_{\mathcal{L}}]$  geometrically

Back to B-model on X

$Q : \mathcal{S}_-(X) \longrightarrow \mathcal{S}_+(X) \oplus \mathcal{S}_-(X)$

$\omega^{-1} = \sum_{k \in \mathbb{Z}} \delta_\Delta t^k \otimes (-t)^{-k+1}$ (as $\omega(-,-) = \mathrm{Tr}(-,-)\,\mathrm{Res}_{t=0}(-,-)$)

where δ_Δ is distributional elt. of $PV(X) \otimes PV(X)$

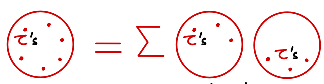
δ_Δ is integral kernel for $1 : PV(X) \longrightarrow PV(X)$

$$P = (\pi_+ \otimes \pi_+)((Q \otimes 1)\omega^{-1}) \underset{(Q = \bar{\partial} + t\partial)}{=\!=\!=} (\partial \otimes 1)\delta_\Delta \in (PV(X) \otimes PV(X))_{\text{distrib.}}$$

Given a local fcl $I(\mu_k) = \int \cdots$ on $\mu = \mu_0 + t\mu_1 + \cdots \in \mathcal{S}_+(X) = PV(X)[\![t]\!]$

$$\delta I = \sum_k \mathrm{Tr}\left(\delta\mu_k \frac{\delta I}{\delta \mu_k}\right)$$

BV bracket $\{I_1, I_2\}_0 \triangleq \mathrm{Tr}\left(\frac{\delta I_1}{\delta \mu_0}\right) \partial \left(\frac{\delta I_2}{\delta \mu_0}\right)$ local

Theorem (Costello-Li) $I_0^X(\mu) = \mathrm{Tr}\langle e^\mu \rangle_0$ 

s.t. $Q I_0^X + \frac{1}{2}\{I_0^X, I_0^X\}_0 = 0$ CME ($\Longleftrightarrow$ topo. recursion relation

So $Q\mu + \sum_{k \geq 2} \frac{1}{k!} l_k(\mu^{\otimes k}) = 0$ where $\delta = Q + \{I_0^X, -\} = \sum_{k \geq 1} l_k$

§ 8 Quantum BV master equation

<u>Toy model</u> (V, ω, Q) dg sympl. v.s.

w/ Lagrangian splitting $V = V_+ \oplus V_-$ $\rightsquigarrow$ $Q : \begin{cases} V_+ \longrightarrow V_+ \\ V_- \longrightarrow V_+ \oplus V_- \end{cases}$

$P = (\pi_+ \otimes \pi_+)\big((Q \otimes 1)\,\omega^{-1}\big) \in \mathrm{Sym}^2 V_+$

$(\mathcal{O}(V_+), Q, \partial_P, \{-,-\}_P)$ dGBV-alg.

Def. (formal) Weyl algebra $\mathcal{W}(V) := \prod_{n \geq 0} (V^*)^{\otimes n}[[\hbar]] \big/ \underbrace{a \otimes b - (-1)^{|a||b|} b \otimes a}_{[a,b]} - \hbar\, \omega^{-1}(a \otimes b)$

$\hbar$: formal quantum parameter

Note: $\mathcal{W}(V)/\hbar\,\mathcal{W}(V) = \hat{\mathcal{O}}(V)$

$\mathrm{Ann}(V_+) := \{\varphi \in V^* \mid \varphi(V_+) = 0\}$, annilator of V_+

$Q : V \longrightarrow V$ preserving V_+

$\overset{\text{dually}}{\Longleftrightarrow}$ $Q : V^* \longrightarrow V^*$ preserving $\mathrm{Ann}(V_+)$

Prop. $\mathrm{Ann}(V_+) \subset V^*$ is a subcomplex

Def. (formal) Fock space $\mathrm{Fock}(V_+) := \mathcal{W}(V) / \mathcal{W}(V)\,\mathrm{Ann}(V_+)$

Remark: Treat $\mathrm{Ann}(V_+)$ as annilation operators.

Q is compatible w/ $\mathcal{W}(V)$

$(\mathcal{W}(V), Q)$ defines a cochain complex

$\mathcal{W}(V)\,\mathrm{Ann}(V_+) \subset \mathcal{W}(V)$ subcomplex

$\rightsquigarrow$ $Q : \mathrm{Fock}(V_+) \longrightarrow \mathrm{Fock}(V_+)$

Lagrangian splitting $V = V_+ \oplus V_- \simeq T^* V_+ \rightsquigarrow V_+^* \subset V^*$

$\hat{\mathcal{O}}(V_+) \triangleq \prod_{n \geq 0} \mathrm{Sym}^n(V_+^*)$ formal fu. on V_+

$$\begin{tikzcd} \hat{\mathcal{O}}(V_+)[[\hbar]] \arrow[r, hook] \arrow[rr, bend right, "\Phi"', "\sim"] & \mathcal{W}(V) \arrow[r, two heads] & \mathrm{Fock}(V_+) \end{tikzcd}$$

creation operators

Q on $\mathcal{W}(V)$ & $\mathrm{Fock}(V_+)$ $\rightsquigarrow$ $\hat{Q}$ on $\hat{\mathcal{O}}(V_+)[[\hbar]]$

$$\begin{tikzcd} \hat{\mathcal{O}}(V_+)[[\hbar]] \arrow[r, "\Phi", "\sim"'] \arrow[d, "\hat{Q}"'] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{Fock}(V_+) \arrow[d, "Q"] \\ \hat{\mathcal{O}}(V_+)[[\hbar]] \arrow[r, "\Phi", "\sim"'] & \mathrm{Fock}(V_+) \end{tikzcd}$$

$$Q : V_+ \longrightarrow V_+ \quad \rightsquigarrow \quad Q : \hat{O}(V_+) \longrightarrow \hat{O}(V_+)$$

Prop. $\hat{Q} = Q + \hbar\, \partial_P$ (std. quant. of quadratic Hamiltonian)

$\hat{Q}^2 = 0$ Pf: exercise.

Quantum Master Eqt. (QME) for $F = \sum_{g \geq 0} F_g \hbar^g \in \hat{O}(V_+)[[\hbar]]$

$$(Q + \hbar\, \partial_P)\, e^{F/\hbar} = 0$$

$$\iff (Q + \hbar\, \partial_P) F + \tfrac{1}{2}\{F, F\}_P = 0$$

$$\xrightarrow{\hbar \to 0} \quad Q F_0 + \tfrac{1}{2}\{F_0, F_0\}_P = 0 \quad \text{(CME)}$$

$$|F\rangle = \Phi(e^{F/\hbar}) \quad (\text{i.e. } e^{F/\hbar}|0\rangle)$$

$$\text{QME} \iff Q|F\rangle = 0 \text{ in } \mathrm{Fock}(V_+)$$

(quantum gauge consistency) $\Longrightarrow$ pass to Q-cohomology.

$$[|F\rangle] \in H^\bullet(\mathrm{Fock}(V_+), Q)$$

Let $\underset{\text{sympl. } \omega}{\mathcal{H} = H^\bullet(V, Q)} \supset \underset{\text{isotropic}}{\mathcal{H}_+ = H^\bullet(V_+, Q)}$

Choose polarizat$^{\underline{n}}$ $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{L}$ Lagr. splitting

$$\begin{array}{ccccc}
W(V) & & \mathrm{Weyl}(\mathcal{H}) & \longleftarrow & \hat{O}(\mathcal{H}_+)[[\hbar^{-}]] \\
\downarrow & \xrightarrow{Q\text{-coh}} & \downarrow & \swarrow \sim & \\
\mathrm{Fock}(V_+) & & \mathrm{Fock}(\mathcal{H}_+) & & \\
Q|F\rangle = 0 & & [|F\rangle]_Q & \longleftrightarrow & e^{F^\alpha/\hbar}
\end{array}$$

$$F^\alpha = \sum_{g \geq 0} F^\alpha_g \hbar^g$$

$F^\alpha_g \in \hat{O}(\mathcal{H}_+)$. quantum invariant.

§9 Homotopic Renormalization (Toy model $(V_+, Q) \rightsquigarrow$) QFT $(\mathcal{E}, Q)$

Space of fields $\mathcal{E} = \Gamma(X, E^\bullet)$ smooth sections of graded bundles

Q (elliptic) differential.

(shifted) Poisson kernel $P \in \Gamma(\mathrm{Sym}^2(\mathcal{E}))_{dist}$ w/ support $\Delta_X \subset X \times X$

$(V_+^* \rightsquigarrow)$ $\mathcal{E}^* = \mathrm{Hom}(\mathcal{E}, \mathbb{C})$ continuous dual

$(V_+^* \otimes V_+^* \rightsquigarrow)$ $\mathcal{E}^* \otimes \mathcal{E}^* = (\mathcal{E} \boxtimes \mathcal{E})^* \supset \mathrm{Sym}^2 \mathcal{E}^*$ etc.

$$\mathcal{O}(\mathcal{E}) = \prod_{n \geq 0} \mathrm{Sym}^n \mathcal{E}^*$$

$(\partial_P : \mathrm{Sym}^n(V_+^*) \to \mathrm{Sym}^{n-2}(V_+^*) \rightsquigarrow)$

$\partial_P : \mathrm{Sym}^n(\mathcal{E}^*) \to \mathrm{Sym}^{n-2}(\mathcal{E}^*)$ NOT well-def$^{\underline{d}}$

Cannot multiply distribut$^{\underline{n}}$! (UV problem in QFT, need renormalizat$^{\underline{n}}$)

Change notation: $P \rightsquigarrow K_0 \in \Gamma(\mathrm{Sym}^2(\mathcal{E}))_{dist}$ (shifted) Poisson kernel

$$Q K_0 := (Q \otimes 1 + 1 \otimes Q) K_0 = 0$$

Key observation: Elliptic regularity $H^\bullet(\text{smooth}, Q) = H^\bullet(\text{distribut}^{\underline{n}}, Q)$

$$\rightsquigarrow \underbrace{K_0}_{\substack{\text{distrib.}\\ \text{kernel}}} = \underbrace{K_r}_{\substack{\text{smooth}\\ \text{kernel}}} + Q(\underbrace{P_r}_{\substack{\text{distribut}^{\underline{n}}.\\ \text{'parametrix'}}})$$

$\partial_{K_r} : \mathrm{Sym}^n(\mathcal{E}^*) \to \mathrm{Sym}^{n-2}(\mathcal{E}^*)$ well-def$^{\underline{d}}$.

$\partial_{K_r} : \mathcal{O}(\mathcal{E}) \longrightarrow \mathcal{O}(\mathcal{E})$ (normalized BV operator).

$[\partial_{K_r}, Q] = 0$ $\quad (\because Q(K_r) = 0)$

Def. $(\mathcal{O}(\mathcal{E}), Q, \partial_{K_r})$ normalized dGBV alg. w.r.t. parametrix P_r.

Given 2 parametrices, $K_0 = K_{r_1} + Q(P_{r_1}) = K_{r_2} + Q(P_{r_2})$

$P_{r_1}^{r_2} := P_{r_2} - P_{r_1} \in \mathrm{Sym}^2(\mathcal{E})$ smooth !

$\rightsquigarrow \partial_{P_{r_1}^{r_2}} : \mathcal{O}(\mathcal{E}) \longrightarrow \mathcal{O}(\mathcal{E})$ 2^{nd} order op., contract w/ $P_{r_1}^{r_2}$

Prop.

$$\begin{tikzcd}
\mathcal{O}(\mathcal{E})[[\hbar]] \arrow[r, "Q + \hbar \partial_{K_{r_1}}"] \arrow[d, "\exp(\hbar \partial_{P_{r_1}^{r_2}})"'] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{O}(\mathcal{E})[[\hbar]] \arrow[d, "\exp(\hbar \partial_{P_{r_1}^{r_2}})"] \\
\mathcal{O}(\mathcal{E})[[\hbar]] \arrow[r, "Q + \hbar \partial_{K_{r_2}}"] & \mathcal{O}(\mathcal{E})[[\hbar]]
\end{tikzcd}$$

i.e. gauge

Pf: $e^{\hbar \partial_{P_{r_1}^{r_2}}} Q e^{-\hbar \partial_{P_{r_1}^{r_2}}} = Q + \hbar [\partial_{P_{r_1}^{r_2}}, Q] + 0 = Q + \hbar(\partial_{K_{r_1}} - \partial_{K_{r_2}})$ #

Def. $\mathcal{O}^+(\mathcal{E})[[\hbar]] := \mathrm{Sym}^{\geq 3}(\mathcal{E}^*) \oplus \hbar\, \mathcal{O}(\mathcal{E})[[\hbar]]$ "action at least cubic modulo $\hbar$"

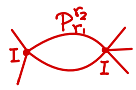
Homotopic renormalization group (HRG)

$$W(P_{r_1}^{r_2}, -) : \mathcal{O}^+(\mathcal{E})[[\hbar]] \longrightarrow \mathcal{O}^+(\mathcal{E})[[\hbar]]$$

$$W(P_{r_1}^{r_2}, I) = \hbar \log\left(e^{\hbar \partial_{P_{r_1}^{r_2}}} e^{I/\hbar}\right)$$

i.e. $$e^{W(P_{r_1}^{r_2}, I)/\hbar} = e^{\hbar \partial_{P_{r_1}^{r_2}}} e^{I/\hbar}$$

$$\rightsquigarrow \quad W(P_{r_1}^{r_2}, I) = \sum_{\Gamma \text{ connected}} W_\Gamma(P_{r_1}^{r_2}, I) \qquad \Gamma$$



I : vertex

$P_{r_1}^{r_2}$: propagator.

Def: A Sol$^{\underline{n}}$ of effective QME is $I[r] \in \mathcal{O}^+(\mathcal{E})[[\hbar]]$ for each P_r s.t.

(1) Renormalized QME, $(Q + \hbar \partial_{K_{r_1}}) e^{I[r]/\hbar} = 0$.

(2) Homotopy RG, $I[r_2] = W(P_{r_1}^{r_2}, I[r_1])$

(Earlier prop. $\Rightarrow$ (1) & (2) are compatible).

§10 Quantum BCOV theory

B-model on X: $\mathcal{E} = S_+(X) = PV(X)[[t]][2]$

shifted Poisson kernel $K_0 = (\partial \otimes 1)\, \delta_\Delta \in PV(X) \otimes PV(X)$

Choose Kähler metric g on $X \rightsquigarrow \bar\partial^* : PV^{i,j} \longrightarrow PV^{i,j-1}$

$\longrightarrow$ heat kernel $h_r^g \in PV(X) \otimes PV(X)$ for $r > 0$

$$\left(e^{-r[\bar\partial, \bar\partial^*]} \alpha\right)(x_1) = \mathrm{Tr}_{x_2}\, h_r^g(x_1, x_2)\, \alpha(x_2)$$

Let $K_r = (\partial \otimes 1)\, h_r^g \xrightarrow{r \to 0} K_0$

Prop: $K_0 = K_r + Q(P_r)$ w/ $P_r = \int_0^r (\bar\partial^* \partial \otimes 1)\, h_u^g\, du$

Pf: This is the integral kernel repr. of the operator eqt.

$$\partial \overset{?}{=} \partial e^{-r[\bar\partial, \bar\partial^*]} + \left[Q, \int_0^r \bar\partial^* \partial\, e^{-u[\bar\partial, \bar\partial^*]} du\right]$$

$$Q = \bar\partial + t\partial, \quad [\bar\partial, \partial] = 0, \quad [\bar\partial^*, \partial] = 0$$

$$\Rightarrow \left[Q, \int_0^r \bar\partial^* \partial\, e^{-u[\bar\partial, \bar\partial^*]} du\right]$$

$$= \int_0^r [\bar\partial, \bar\partial^*]\, \partial\, e^{-u[\bar\partial, \bar\partial^*]} du = \partial - \partial e^{-r[\bar\partial, \bar\partial^*]} \quad \#$$

Def: Given parametrix P_r w/ renormalized Poisson kernel,

effective propagator $P_\varepsilon^L \in \mathrm{Sym}^2(PV(X)) \subset \mathrm{Sym}^2\mathcal{E}$ for $0<\varepsilon<L$

$$P_\varepsilon^L = \int_\varepsilon^L (\bar\partial^* \partial \otimes 1)\, h_u^g \, du \quad \text{smooth!}$$

(homotopy from regularizat^{n} K_ε to K_L)

Def (Costello-Li) perturbation quantization of BCOV theory on X is

$$\forall\, L>0, \quad F[L] = \sum_{g=0}^{\infty} \hbar^g F_g[L] \in \mathcal{O}^+(S_+(X))[\![\hbar]\!] \quad \text{s.t.}$$

(1) homotopy RG flow: $F[L] = W(P_\varepsilon^L, F[\varepsilon])$ (i.e. $e^{F[L]/\hbar} = e^{\hbar \partial_{P_\varepsilon^L}} e^{F[\varepsilon]/\hbar}$)

(2) renormalized QME $(Q + \hbar \Delta_L) e^{F[L]/\hbar} = 0$ ("$\Delta_L = \partial_{K_L}$")

(3) locality axiom: $F[L]$ has small L asym expansion via local functional.

(4) classical limit: $\lim_{L\to 0} F[L] = I_0^X$

(5) degree axiom & Hodge weight axiom: $F_g[L]$ has Hodge wt. $(3-d)(g-1)$

(Hodge wt. of $t^m PV^k$ is $k+m-1$)

IR limit $L \to \infty$,

$$K_L = (\partial \otimes 1) e^{-L[\bar\partial, \bar\partial^*]} \xrightarrow{L\to\infty} (\partial\otimes 1)(\text{harmonic proj.}) = 0$$

$$\Longrightarrow \Delta_L = \partial_{K_L} \xrightarrow{L\to\infty} 0$$

$$(Q+\hbar\Delta_L) e^{F[L]/\hbar} = 0 \Longrightarrow Q\, e^{F[\infty]/\hbar} = 0 \Rightarrow QF[\infty] = 0$$

$$\Longrightarrow [F[\infty]]_\infty \in H^\bullet(\mathcal{O}(S_+(X))[\![\hbar]\!], Q) \simeq \mathcal{O}(\underbrace{H^\bullet(S_+(X), Q)})[\![\hbar]\!]$$

via g & splitting of Hodge filt. $\rightsquigarrow$ $H^\bullet(S_+(X), Q) \simeq H^\bullet(X, \wedge^\bullet T_X)[\![t]\!]$

$\rightsquigarrow F^\alpha_{g,n,X} : \mathrm{Sym}^n(H^\bullet(X, \wedge^\bullet T_X)[\![t]\!]) \longrightarrow \mathbb{C}$ (mirror to GW-invariants)

UV finiteness and locality

In practice, construct quantization

(1) Start w/ local fcl $I \in \mathcal{O}^{+}_{loc}(\mathcal{E})[[\hbar]]$ (classical interaction)

(2) Find ϵ-indep. local fcl $I^{CT}(\epsilon) \in \hbar\, \mathcal{O}_{loc}(\mathcal{E})[[\hbar]]$ (counter terms)

s.t. $e^{I[L]/\hbar} := \lim_{\varepsilon \to 0} e^{\hbar \partial_{P^L_\varepsilon}} e^{(I + I^{CT}(\epsilon))/\hbar}$ exists ($\rightsquigarrow \{I[L]\}_L$ s.t. HRG)

(3) Further correction to solve QME.
(could be obstructed — gauge anomaly)

Def: UV finite if $\lim_{\varepsilon \to 0} e^{\hbar \partial_{P^L_\varepsilon}} e^{(I + \cancel{I^{CT}(\epsilon)})/\hbar}$ exists.

Assume UV finite

$$\hbar \Delta I + QI + \tfrac{1}{2}\{I, I\} = 0 \quad \text{ill def}^{d} \text{ at } L = 0$$

$$\xrightarrow{\text{regularize}} \hbar \Delta_L I[L] + Q I[L] + \tfrac{1}{2}\{I[L], I[L]\} = 0 \quad \text{well-def}^{d} \text{ for } L > 0$$

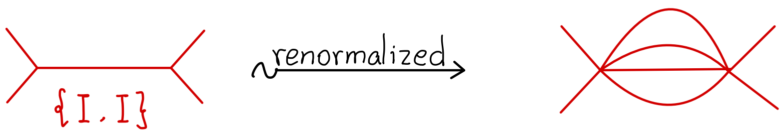
$$\xrightarrow{L \to 0} QI + \tfrac{1}{2}\{I, I\} + \ell_3(I, I, I) + \dots = 0$$

$\rightsquigarrow$ renormalized local QME

- explicit formulae for ℓ_k's not known.

<u>Thm</u> (Li) 2d Chiral theories are UV finite.

Renormalized local QME $\Longleftrightarrow$ MC eqt. in Chiral VOA



CME

ref: Vertex alg. and quantum master eqt.

- Explicit canonical sol^{n} of quantum B-model on elliptic curves and establish quantum mirror conjecture.

<u>Thm</u> (Wang-Yan) Chiral theories of any dim. are UV finite.

e.g. BCOV, HCS $\longrightarrow$ quantum BCOV has a local expression.

(Gui) Higher chiral OPE.

§11 Landau-Ginzburg B-model

$f:(X,\Omega) \xrightarrow{\text{holo. fu.}} \mathbb{C}$, $\quad \mathrm{Crit}(f)$ compact

cpx. mfd. $\quad$ holo. vol. form

$PV(X) = \Omega^{0,*}(X, \wedge^\bullet T_X) \supset PV_c(X)$ compact support

$\bar\partial_f = \bar\partial + \{f, -\} = \bar\partial + df \lrcorner$ & $\partial = \partial_\Omega$

$(PV_c(X), \bar\partial_f, \partial) \subset (PV(X), \bar\partial_f, \partial)$ dGBV

Lemma $(PV_c(X), \bar\partial_f) \subset (PV(X), \bar\partial_f)$ quasi-isom.

[Pf: $\{f,-\}$ acyclic outside a small nbd of $\mathrm{Crit}(f)$ #

To apply Barannikov-Kontsevich constr. for Frobenius mfd. str. on $H^\bullet(PV, \bar\partial_f)$, need (1) Hodge-to-deRham E_1-degen. of spectral seq. for t-adic filt of

$(PV(X)[\![t]\!], \bar\partial_f + t\partial)$ (or $(PV_c(X)[\![t]\!], \bar\partial_f + t\partial)$)

($\rightsquigarrow$ smoothness, formality, hence a univ. period map

(2) Trace pairing $\mathrm{Tr}(-,-)$ on PV, compat. w/ $\bar\partial_f$ & ∂.

$\longrightarrow$ sympl. str. $\omega = \mathrm{Tr}(-,-)\,\mathrm{Res}_{t=0}$

(3) Splitting of $H^\bullet(PV, \bar\partial_f) \longrightarrow H^\bullet(PV_c(X)[\![t]\!], \bar\partial_f + t\partial)$

compat. w/ Tr (and higher residue) $\rightsquigarrow$ "good basis".

Remark: $\mathrm{Crit}(f)$ isolated $\Rightarrow H^\bullet(PV, \bar\partial_f) = \mathrm{Jac}(f) = \mathcal{O}(\mathrm{Crit}(f))$

Hodge-to-deRham E_1-degen. ✓

Trace pairing $\quad \mathrm{Tr}: PV_c(X) \longrightarrow \mathbb{C} \qquad \mathrm{Tr}(\mu) = \int_X (\mu \lrcorner \Omega) \wedge \Omega$

Prop: $\mathrm{Tr}((\bar\partial_f \alpha)\beta) = -(-1)^{|\alpha|}\mathrm{Tr}(\alpha\, \bar\partial_f \beta)$; $\mathrm{Tr}((\partial\alpha)\beta) = (-1)^{|\alpha|}\mathrm{Tr}(\alpha\,\partial\beta)$

Cor. Tr defines a pairing on cohomologies.

$\mathrm{Tr}: H^\bullet(PV_c, \bar\partial_f) \otimes H^\bullet(PV_c, \bar\partial_f) \longrightarrow \mathbb{C}$

More generally, $K_f: H^\bullet(PV_c[\![t]\!], Q_f) \otimes H^\bullet(PV_c[\![t]\!], Q_f) \longrightarrow \mathbb{C}[\![t]\!]$

$K_f(\alpha f(t), \beta g(t)) = \mathrm{Tr}(\alpha\beta) f(t) g(-t) \qquad Q_f = \bar\partial_f + t\partial$

Eg. $f: (\mathbb{C}^n, \Omega = d^n z) \longrightarrow \mathbb{C}$ w/ $\mathrm{Crit}(f) = \{0\}$

$H^\bullet(PV_c, \bar\partial_f) = \mathrm{Jac}(f) \xlongequal{\Omega} \Omega^n_{hol} / df \wedge \Omega^{n-1}_{hol} =: \Omega_f$

$\Longrightarrow$ Viewing Tr on Ω_f, $\mathrm{Tr}(\alpha(z) d^n z, \beta(z) d^n z) \xlongequal{\text{Li-Li-Saito}} \mathrm{Res}_{z=0} \dfrac{\alpha\beta\, d^n z}{\partial_1 f \wedge \cdots \wedge \partial_n f}$

& $K_f(\alpha(z), \beta(z)) = \underset{\text{Res}}{K_0(\alpha,\beta)} + \underset{\text{higher residues of K. Saito}}{K_1(\alpha,\beta)} t + \cdots\cdots$

Hodge-to-deRham : (i) PV too big for Tr (ii) PV_c too small for Hodge decomp.
(iii) PV_{L^2} 'bad' for nonlinear str., eg. product.

(Li-Wen) $PV_c(X) \subset PV_{f,\infty}(X) \subset PV(X)$ w/ $PV_{f,\infty}(X)$ solves (i) (ii) (iii).

$PV_{f,\infty}(X) := \{ \mu \in PV_{L^2}(X) \mid |\nabla f|^i \nabla^j \mu \in L^2 \quad \forall\, i,j \}$

Assume (X, g, Ω) Kähler w/ bounded CY geometry.

Def: $f : X \xrightarrow{\text{holo.}} \mathbb{C}$ 'strong elliptic' if $\forall \varepsilon > 0, \forall k \geq 2$, $\varepsilon |\nabla f|^k - |\nabla^k f| \xrightarrow{\text{as } z \to \infty} +\infty$

Eg. $(\mathbb{C}^n, d^n z)$, g_{std}, f non-degen. quasi-homog. polyn.
($\sim$ mirror of FJRW theory)

Eg. $(\mathbb{C}^{*n}, \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n})$, g_{std}, f convenient non-degen. Laurent polyn. ($\sim$ mirror of toric)

Eg. Crepant resol$^{\underline{n}}$ of $\mathbb{C}^n/\Gamma, d^n z$ w/ $\Gamma \overset{\text{finite}}{\leq} SU(n)$, g_{ALE} (orbifold LG B-model)

<u>Thm</u> (Li-Wen) $f : (X, g, \Omega) \xrightarrow{\text{holo.}} \mathbb{C}$, strong elliptic, cpt. $\mathrm{Crit}(f)$
bounded CY geometry

(1) $(PV_{f,\infty}(X), \bar\partial_f, \partial)$ dGBV, w/ pairing Tr, Hodge-to-deRham degen. ✓

(2) $(PV_c(X), \bar\partial_f) \subset (PV_{f,\infty}(X), \bar\partial_f) \subset (PV(X), \bar\partial_f)$ quasi-isom.
($\Longrightarrow$ Hodge-to-deRham degen. ✓ for all)

(3) Poincaré duality
$\mathrm{Tr} : H^\bullet(PV_{f,\infty}(X), \bar\partial_f) \otimes H^\bullet(PV_{f,\infty}(X), \bar\partial_f) \longrightarrow \mathbb{C}$ non-degen.
($\rightsquigarrow$ K_f generalize K. Saito higher residue)

Cor. Frobenius str. on $H^\bullet(PV(X), \bar\partial_f)$.

Remark: $\exists$ char. p approach.

§12 Open-closed B-model

closed string field B-model $\Longrightarrow$ BCOV theory

[Witten 1992
open string field B-model $\Longrightarrow$ CS + instantons
open string field A-model $\Longrightarrow$ Holom. CS

holo. v.b. $E \longrightarrow (X, \Omega_X)$ CY3

fields $\mathcal{E} = \Omega^{0,*}(X, \mathrm{End}\, E)[1] \ni A = \underset{\text{ghost}}{A} + \underset{\text{physical field}}{A} + \underset{\text{anti-ghost}}{A} + \underset{\text{anti-field}}{A}$

HCS action $HCS(A) = \int_X \mathrm{Tr}_E \left(\frac{1}{2} A \wedge \bar\partial A + \frac{1}{6} A \wedge [A, A] \right) \wedge \Omega_X$

EOM, for A_1, $\bar\partial A_1 + \frac{1}{2}[A_1, A_1] = 0$

$\bar\nabla = \bar\partial + A_1$, $(\bar\nabla)^2 = 0 \rightsquigarrow$ deform holo. v.b. str. on E.

HCS describes moduli of holom. v.b.

$\mathcal{E}$: (-1)-shifted sympl. w/ $\omega(\alpha, \beta) = \int_X \mathrm{Tr}(\alpha \wedge \beta) \wedge \Omega_X$

$\omega^{-1} \rightsquigarrow$ shifted Poisson kernel $\delta_\Delta \in (\mathcal{E} \otimes \mathcal{E})_{\mathrm{dist.}}$

BV bracket $\{I_1, I_2\}_0$ for local fcl I's.

HCS satisfies CME, $\{HCS, HCS\}_0 = 0$

Write $\delta_{HCS} := \{HCS, -\}_0$, $(\delta_{HCS})^2 = 0$ BRST-transf.

$\delta_{HCS} A = \bar\partial A + \frac{1}{2}[A, A]$, $\rightsquigarrow$ dgLa $\Omega^{0,*}(X, \mathrm{End}\, E)$

Large N perspective: $E \xhookrightarrow{\text{subbdl.}} L^{\oplus N}$, L ample enough

$\longleftrightarrow$ idempotent $P \in \mathfrak{gl}_N(C^\infty(X))$, $P^2 = P$, $E = P(L^{\oplus N})$

$\Omega^{0,*}(X, \mathrm{End}\, E) \hookrightarrow \Omega^{0,*}(X) \otimes \mathfrak{gl}_N = \mathcal{E}_N$

$\mathcal{E}_N \subset \mathcal{E}_{N+1} \subset \mathcal{E}_{N+2} \subset \cdots\cdots$

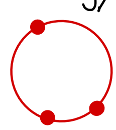
Look for fcl. on $\mathcal{E}_N$, stable as $N \to \infty$, called "admissible fcl". Eg. $\int_X \mathrm{Tr} A^3 \wedge \Omega_X$

<u>Large N duality</u> HCS on $\mathcal{E}_N$, $J = \delta(HCS)$ 1st order deformat$^{\underline{n}}$, loc. adm.

$\xRightarrow{\text{CME}} \delta_{HCS} J = \{HCS, J\}_0 = 0 \Longrightarrow J$ in loc. dual of $H^{\mathrm{Lie}}_\bullet(\mathfrak{gl}_N(\Omega^{\bullet,\bullet}(X)), \bar\partial, [-,-])$

(Loday-Quillen-Tsygan) $\lim\limits_{N \to \infty} H^{\mathrm{Lie}}_\bullet(\mathfrak{gl}_N(R)) = \mathrm{Sym}(HC_{\bullet-1}(R))$ cyclic homology

$HC_\bullet(\Omega^{\bullet,\bullet}(X)) \overset{HKR}{=\!=} (\underbrace{\Omega^{\bullet,\bullet}(X)[t^{-1}]}_{\Omega^{\bullet,\bullet}(X)((t^{-1}))/t\,\Omega^{\bullet,\bullet}(X)[[t^{-1}]]}, \bar\partial + t\partial)$ $\quad (HC_\bullet(R) \sim (C_\bullet(R)[t^{-1}], b + tB))$



$(\Omega^{\bullet,\bullet}(X)[t^{-1}], \bar\partial + t\partial) \xleftrightarrow{\text{local dual}} (PV(X)[\![t]\!], Q=\bar\partial + t\partial) \longrightarrow S_+$

BCOV field (gravity) $=$ single trace operator at $N\to\infty$

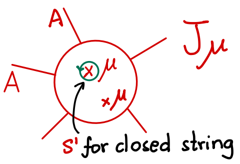
i.e Gauge / Gravity duality

Eg 1. $\mu \in PV^{k,\bullet}(X) \longrightarrow 1^{st}$ order deformat$^{\underline{n}}$.

$$J_\mu = \int_X \mathrm{Tr}\,(\mu \lrcorner A \wedge \partial A \wedge \cdots \wedge \partial A)\wedge \Omega_X \qquad (\mathrm{HKR})$$

Eg 2. $\mu \in t^m PV^{k,\bullet}(X) \longrightarrow 1^{st}$ order deformat$^{\underline{n}}$.

$$J_\mu = \sum_{l_1+\dots+l_k=2m}\int_X \mathrm{Tr}\,(\mu \lrcorner A \wedge A^{l_1}\wedge\partial A \wedge A^{l_2}\wedge\partial A\wedge \cdots \wedge A^{l_k}\wedge \partial A)\wedge\Omega_X$$



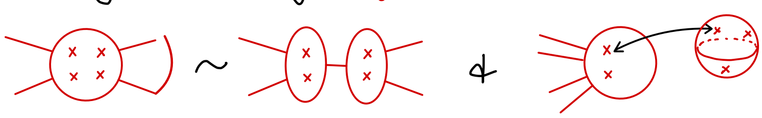
J_μ general formula for higher order deformat$^{\underline{n}}$ at the disk level by Willwacher-Calaque's cyclic ext$^{\underline{n}}$ of Kontsevich formality.

$\int_{\mathrm{Conf}_{n,m}(D^2)} \longrightarrow$ Poisson σ-model.

<u>Open-closed Master Eqt</u> for $I_{g,h}(\mu, A)$ $\quad g$: genus, h: # bdy. comp.

$\{I_{g,h}(\mu,A)\}$ needs to s.t. Zviebech's open-closed master eqt.

$\sim \quad \partial(\mathcal{M}_{\Sigma_{g,h}}) \sim$ degenerat$^{\underline{n}}$ of $\Sigma_{g,h}$

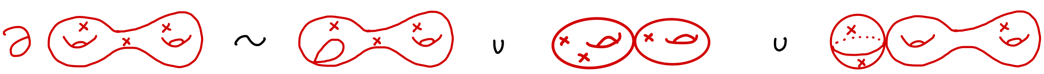
Eg $(g,h)=(0,1)$, $\partial($  $)$

$$Q I_{0,1} + \underbrace{\tfrac{1}{2}}_{\text{(symmetry)}}\{I_{0,1}, I_{0,1}\}_o + \{I_{0,1}, I_{0,0}\}_c = 0$$

open BV bracket $\quad$ closed BV bracket

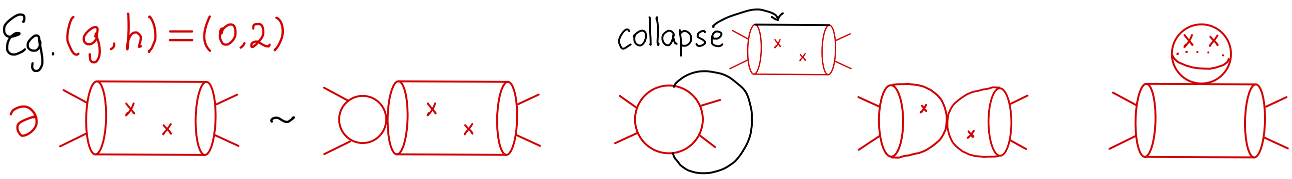
$\sim$ Kontsevich formality $\{-, I_{0,0}\}_c$.

Eg. $(g,h)=(2,0)$ (i.e. $(Q+\hbar\Delta_c)I + \frac{1}{2}\{I,I\}_c = 0$)



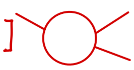
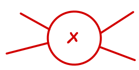
$$Q I_{2,0} + \Delta_c I_{1,0} + \tfrac{1}{2}\{I_{1,0}, I_{1,0}\}_c + \{I_{0,0}, I_{2,0}\}_c = 0$$

Eg. $(g,h)=(0,2)$



$$Q I_{0,2} + \{I_{0,0}, I_{0,2}\}_o + \underbrace{\Delta_o I_{0,1}}_{\text{1-loop anomaly}} + \tfrac{1}{2}\{I_{0,1}, I_{0,1}\}_c + \{I_{0,0}, I_{0,2}\}_c = 0$$

$\sim \mathrm{Tr}_{\mathfrak{gl}_N}(\bigcirc) = N \neq 0$ $\left(\mathrm{Tr}_{\mathfrak{gl}_{N|N}}(\bigcirc) = 0 \ \begin{smallmatrix}\sim \text{virtual VB} \\ \text{in K-th., rk } 0\end{smallmatrix}\right)$

<u>Thm</u> (Costello-Li) HCS on $\Omega^{0,\bullet}(\mathbb{C}^n)\otimes\mathfrak{gl}_{N|N}[1]$  w/ 1^{st} order deformat$^{\underline{n}}$ 

$\exists$ canon. quantizat$^{\underline{n}}$ $I_{g,h}$'s s.t. Zviebech's open-closed master eqt.

Remark: Such quantizat$^{\underline{n}}$ glue to local CY w/ nontrivial $\mathbb{C}^\times$-action.

Remark: BCOV + HCS open-closed B-model is UV finite $\forall$ CY.